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Sem - VI
Paper DSE - III
Theory of Equations.

(1) (i) Write polynomial equations.

Ans. The general form of polynomial eqn of nth degree is

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n = 0$$

Where $a_0, a_1, a_2, \ldots a_n$ are constants.

The exponent of the highest power of x in a polynomial determines degree of that polynomial equation.

(ii) Define symmetric function.

Ans. If an expression involving the roots of an equation remain unaltered when any two of the roots are interchanged, then such an expression is called a symmetric function of the roots of the given eq^n.

(iii) Define Reciprocal equations.

Ans. If an equation remains unchanged when x is changed into its reciprocal, the such an equation is called reciprocal equation.

(iv) Define the discriminant of the cubic.

Ans: the discriminant of an equation is a ~~quantity~~ quantity, the vanishing of which expresses the condition for equal roots.

(x) write Euler's cubic

Ans: Let p, q, r are the roots of the cubic

$$t^3 + 3Ht^2 + \left(3H^2 - \frac{a_0^2 I}{4}\right)t - \frac{G^2}{4} = 0$$

which is called the Euler's cubic.

(vi) state Descarte's Rule of signs:

Ans: In the equation $f(x) = 0$

(i) the number of positive roots cannot exceed the number of changes of sign in $f(x)$.

(ii) The number of negative roots cannot exceed the number of changes of sign in $f(-x)$.

(vii) Define superior or upper limit of positive roots

Ans: A number which is greater than all the positive roots of the given equation is called an upper limit of the positive roots of the eqn.

(viii) Define lower or inferior limit of the positive roots

Ans: A number which is less than all the positive roots of a given equation is called an lower limit of the positive roots of that equation.

Theorem: Prove that Every equation of nth degree has n roots and no more.

Proof. Let α_1 be the root of $f_n(x)=0$, then $f_n(x)$ is divisible by $(x-\alpha_1)$ without remainder.

$\therefore f_n(x) \equiv (x-\alpha_1) f_{n-1}(x)$ ------ (1)

Where $f_{n-1}(x)$ the quotient is a polynomial of $(n-1)^{th}$ degree in x.

Again, according to the fundamental theorem, $f_{n-1}(x)=0$ has a root; let the root be α_2

Then $f_{n-1}(x)$ is divisible by $(x-\alpha_2)$

$\therefore f_{n-1}(x) \equiv (x-\alpha_2) f_{n-2}(x)$

and consequently from (1), $f_n(x) = (x-\alpha_1)(x-\alpha_2) f_{n-2}(x)$

Illy $f_{n-2}(x)=0$ has a root, α_3 (say)

then we have $f_{n-2}(x) = (x-\alpha_3) f_{n-3}(x)$

Thus, $f_n(x) \equiv (x-\alpha_1)(x-\alpha_2)(x-\alpha_3) f_{n-3}(x)$

Continuing this process, we get

$f_n(x) \equiv (x-\alpha_1)(x-\alpha_2)(x-\alpha_3) \ldots (x-\alpha_n)\, Q$ --- (2)

Now each of $f_n(x)$ and $(x-\alpha_1)(x-\alpha_2) \ldots (x-\alpha_n)$ is of nth degree and hence Q must be independent of x.

Now $f_n(x) \equiv a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1}x + a_n$

equation the coefficient of x^n from the eqn (2)

we get $Q = a_0$

Thus from (2), $f_n(x) \equiv a_0 (x-\alpha_1)(x-\alpha_2)(x-\alpha_3) \ldots (x-\alpha_n)$ -- (3)

Now, the R.H.S. of (3) vanishes when $x = \alpha_1, \alpha_2, \ldots \alpha_n$

The eqn $f_n(x)=0$ has, therefore n roots.

Now to prove that $f_n(x)=0$ has got n and only n roots.

Let $\delta \neq \alpha_1, \alpha_2, \ldots \alpha_n$, Then no factors of $f_n(x)$ can vanish as is evident from (3) & consequently $f_n(x) \neq 0$ for $x=\delta$. Hence $f_n(x)=0$ cannot more than n roots.

This proves the theorem.

Q. One of the roots of the eqn $x^3+x^2-x+15=0$ is -3. Find the other roots.

Soln. Since $x=-3$ is a root of the eqn $x^3+x^2-x+15=0$

$\therefore$ $x+3$ must be a factor of x^3+x^2-x+15.

Now, dividing x^3+x^2-x+15 by $x+3$,

we get x^2-2x+5

That is $x^3+x^2-x+15=(x+3)(x^2-2x+5)$

Thus the order roots of the eqn are obtained by solving $x^2-2x+5=0$

Now, from $x^2-2x+5=0$, we get

$$x=\frac{2\pm\sqrt{4-4\cdot1\cdot5}}{2}=\frac{2\pm 4i}{2}=1+2i$$

Hence the other roots of the given eqn are

$1+2i$, $1-2i$ Ans.

Que; Solve the eqn $2x^3-15x^2+37x-30=0$ whose roots are in A.P..

Soln. Let $\alpha-\delta, \alpha, \alpha+\delta$ be the roots of the given equation.

Then, $(\alpha-\delta)+\alpha+(\alpha+\delta)=\frac{15}{2} \Rightarrow 3\alpha=15/2 \quad \therefore \alpha=5/2$

Also, $(\alpha-\delta)\alpha+(\alpha-\delta)(\alpha+\delta)+\alpha(\alpha+\delta)=\frac{37}{2}$

$\Rightarrow \alpha^2-\cancel{\alpha\delta}+\alpha^2-\delta^2+\alpha^2+\cancel{\alpha\delta}=\frac{37}{2}$

$\Rightarrow 3\alpha^2-\delta^2=\frac{37}{2} \Rightarrow 3\cdot\frac{25}{4}-\delta^2=\frac{37}{2}$

$\Rightarrow \delta^2=\frac{75}{4}-\frac{37}{2}=\frac{1}{4} \quad \therefore \delta=\pm\frac{1}{2}$

$\therefore$ The roots are $\frac{5}{2}-\frac{1}{2}, \frac{5}{2}, \frac{5}{2}+\frac{1}{2}$

i,e $2, \frac{5}{2}, 3$ Ans.

Q. Solve the equation $6x^3-11x^2+6x-1=0$ whose roots are in H.P.

Soln. Let the roots of the eqn $6x^3-11x^2+6x-1=0$ be α, β, γ.

Then $\alpha+\beta+\gamma = \frac{11}{6}$ — (1), $\alpha\beta+\beta\gamma+\gamma\alpha = 1$ — (2), $\alpha\beta\gamma = \frac{1}{6}$ — (3)

Also α, β, γ are in H.P.

Hence β is the H.M. betn α & β and $\therefore \beta = \frac{2\alpha\gamma}{\alpha+\gamma}$

$\Rightarrow \beta(\alpha+\gamma) = 2\alpha\gamma \Rightarrow \beta\alpha+\beta\gamma = 2\alpha\gamma$

$\Rightarrow \alpha\beta+\beta\gamma+\gamma\alpha = 3\gamma\alpha$ —— (4)

From 2 & 4, $3\gamma\alpha = 1 \Rightarrow \gamma\alpha = \frac{1}{3}$ —— (5)

Hence from (3), $\beta \cdot \frac{1}{3} = \frac{1}{6} \Rightarrow \beta = \frac{1}{2}$

From (1), $\alpha+\frac{1}{2}+\gamma = \frac{11}{6} \Rightarrow \alpha+\gamma = \frac{11}{6}-\frac{1}{2} = \frac{4}{3}$ —— (6)

We now solve (5) & (6) for α and γ

We have $(\alpha-\gamma)^2 = (\alpha+\gamma)^2 - 4\alpha\gamma = \frac{16}{9}-\frac{4}{3} = \frac{4}{9}$

$\therefore \alpha-\gamma = \pm\frac{2}{3}$ —— (7)

Solving (6) & (7), we get $\alpha = 1, \gamma = \frac{1}{3}$ or $\alpha = \frac{1}{3}, \gamma =$

Hence in either case,

the roots are $1, \frac{1}{2}$ and $\frac{1}{3}$

Que: Find the condition that the roots of the eqⁿ
$ax^3 + 3bx^2 + 3cx + d = 0$ are in
(i) A.P. (ii) G.P. (iii) H.P.

Solⁿ: (i) Let $\alpha-\delta, \alpha, \alpha+\delta$ be the roots of the given eqⁿ
Then $(\alpha-\delta)+\alpha+(\alpha+\delta) = -\frac{3b}{a} \Rightarrow 3\alpha = -\frac{3b}{a}$ ∴
$\therefore \alpha = -\frac{b}{a}$ —— (1)

Now, since α is a root of the eqⁿ $ax^3+3bx^2+3cx+d$
∴ we have $a\alpha^3+3b\alpha^2+3c\alpha+d=0$
hence substituting the value of α from (1), we get
$$a\left(-\frac{b}{a}\right)^3 + 3b\left(-\frac{b}{a}\right)^2 + 3c\left(-\frac{b}{a}\right) + d = 0$$
$$\Rightarrow -\frac{b^3}{a^2} + \frac{3b^3}{a^2} - \frac{3bc}{a} + d = 0$$
$$\Rightarrow \frac{2b^3}{a^2} - \frac{3bc}{a} + d = 0 \Rightarrow 2b^3 - 3abc + a^2d = 0$$
Which is required condition.

(ii) We have, product of the roots $\alpha\beta\gamma = -\frac{d}{a}$ —(1)
Since α, β, γ are in G.P. we may take them
as $\alpha = \frac{\lambda}{\gamma}$, $\beta = \gamma$ & $\gamma = \lambda\gamma$

∴ from (1), $\frac{\lambda}{\gamma}\cdot\lambda\cdot\lambda\gamma = -\frac{d}{a}$ i.e $\lambda^3 = -\frac{d}{a}$ - -

Since $\lambda (=\beta)$ is a root of the given eqⁿ,
∴ we have $a\lambda^3+3b\lambda^2+3c\lambda+d=0$
$\Rightarrow a\left(-\frac{d}{a}\right)+3b\lambda^2+3c\lambda+d=0$ since $\lambda^3 = -\frac{d}{a}$
$\Rightarrow -d+3b\lambda^2+3c\lambda+d \Rightarrow 3b\lambda^2+3c\lambda=0$
$\Rightarrow b\lambda+c=0 \Rightarrow b\lambda=-c \Rightarrow b^3\lambda^3=-c^3$
$\Rightarrow b^3\left(-\frac{d}{a}\right)=-c^3 \Rightarrow b^3d=ac^3 \Rightarrow b^3d-c^3a=0$
Which is required condition.

(iii) Let the roots be α, β, γ.

Then $\alpha+\beta+\gamma = -\frac{3b}{a}$ --- (1)

$\alpha\beta+\beta\gamma+\gamma\alpha = \frac{3c}{a}$ --- (2)

$\alpha\beta\gamma = -d/a$ --- (3)

Also, since α, β, γ are in H.P. $\therefore \frac{1}{\alpha}+\frac{1}{\gamma} = \frac{2}{\beta}$ ---

from (4) we have $\frac{\alpha+\gamma}{\alpha\gamma} = \frac{2}{\beta}$ i.e. $\alpha\beta+\beta\gamma = 2\alpha$

$\Rightarrow \alpha\beta+\beta\gamma+\gamma\alpha = 3\gamma\alpha$

$\Rightarrow \frac{3c}{a} = 3\gamma\alpha$ from eqn (2)

$\therefore \gamma\alpha = c/a$

hence from (3), $\beta \cdot c/a = -\frac{d}{a}$ $\therefore \beta = -\frac{d}{c}$

Now, since β is the roots of the eqn

$ax^3+3bx^2+3cx+d=0$

$\therefore$ we have ~~$a\beta^3 + 3b$~~

$a\beta^3+3b\beta^2+3c\beta+d=0$

$\Rightarrow a\left(-\frac{d}{c}\right)^3+3b\left(-\frac{d}{c}\right)^2+3c\left(-\frac{d}{c}\right)+d=0$

$\Rightarrow -\frac{ad^3}{c^3}+\frac{3bd^2}{c^2}-2d=0$

$\Rightarrow -ad^3+3bcd^2-2c^3d=0$

$\Rightarrow -ad^2+3bcd-2c^3=0$

$\Rightarrow ad^2-3bcd+2c^3=0$

Which is the required condition

Que: Find the condition that the roots of the biquadrati[c] $x^4+px^3+qx^2+rx+s=0$ may be in G.P.

Solve: Let the roots of the biquadratic eqⁿ be $\alpha/\rho^3,\ \frac{\alpha}{\rho},\ \alpha\rho,\ \alpha\rho^3$

Then we have $\frac{\alpha}{\rho^3}+\frac{\alpha}{\rho}+\alpha\rho+\alpha\rho^3=-p$

$\Rightarrow \alpha\left(\frac{1}{\rho^3}+\frac{1}{\rho}+\rho+\rho^3\right)=-p$ --- (1)

and $\frac{\alpha}{\rho^3}\cdot\frac{\alpha}{\rho}+\frac{\alpha}{\rho^3}\cdot\alpha\rho+\frac{\alpha}{\rho^3}\cdot\alpha\rho^3+\frac{\alpha}{\rho}\cdot\alpha\rho+\frac{\alpha}{\rho}\cdot\alpha\rho^3+\alpha\rho\cdot\alpha\rho^3=q$

$\Rightarrow \frac{\alpha^2}{\rho^4}+\frac{\alpha^2}{\rho^2}+\alpha^2+\alpha^2+\alpha^2\rho^2+\alpha^2\rho^4=q$

$\Rightarrow \alpha^2\left(\frac{1}{\rho^4}+\frac{1}{\rho^2}+2+\rho^2+\rho^4\right)=q$ -- (2)

also $\frac{\alpha}{\rho^3}\cdot\frac{\alpha}{\rho}\cdot\alpha\rho+\frac{\alpha}{\rho^3}\cdot\frac{\alpha}{\rho}\cdot\alpha\rho^3+\frac{\alpha}{\rho^3}\cdot\alpha\rho\cdot\alpha\rho^3+\frac{\alpha}{\rho}\cdot\alpha\rho\cdot\alpha\rho^3=-r$

$\Rightarrow \frac{\alpha^3}{\rho^3}+\frac{\alpha^3}{\rho}+\alpha^3\rho+\alpha^3\rho^3=-r$

$\Rightarrow \alpha^3\left(\frac{1}{\rho^3}+\frac{1}{\rho}+\rho+\rho^3\right)=-r$ -- -- (3)

and $\frac{\alpha}{\rho^3}\cdot\frac{\alpha}{\rho}\cdot\alpha\rho\cdot\alpha\rho^3=s \Rightarrow \alpha^4=s$ -- (4)

Dividing (3) by (1) we get. $\alpha^2=\frac{r}{p}$ $\therefore \alpha^4=\frac{r^2}{p^2}$

i.e $s=\frac{r^2}{p^2}$, from eq (4)

$\therefore r^2-p^2s=0$, which is the required condition.

Soln Calculate the values of following symmetric for the cubic eqn $x^3+px^2+qx+r=0$ whose roots are α, β, γ

(i) $\Sigma\alpha^2$ (ii) $\Sigma\alpha^2\beta^2$, (iii) $\Sigma\alpha^2\beta$ (iv) $\Sigma\alpha^3\beta$

(v) $\Sigma\alpha^3$ (vi) $\Sigma\alpha^4$ (vii) $\Sigma\alpha^2\beta^2$ (viii) $\Sigma\frac{\beta^2+\gamma^2}{\beta\gamma}$

Soln Since α, β, γ are the roots of the eqn $x^3+px^2+qx+r=0$, We have

$\Sigma\alpha = \alpha+\beta+\gamma = -p$, $\Sigma\alpha\beta = \alpha\beta+\beta\gamma+\gamma\alpha = q$ & $\alpha\beta\gamma =$

(i) $\Sigma\alpha^2 = (\alpha+\beta+\gamma)^2 = \Sigma\alpha^2 + 2\Sigma\alpha\beta$

$\therefore \Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta = (-p)^2 - 2q = \underline{p^2-2q}$

(ii) $(\Sigma\alpha\beta)^2 = (\alpha\beta+\beta\gamma+\gamma\alpha)^2$

$= (\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2) + 2\alpha\beta^2\gamma + 2\alpha^2\beta\gamma + 2\alpha\beta\gamma^2$

$= \Sigma\alpha^2\beta^2 + 2\alpha\beta\gamma(\alpha+\beta+\gamma)$

$= (\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma\,\Sigma\alpha$

$= q^2 - 2(-r)(-p) = q^2 - 2pr$

(iii) $\Sigma\alpha\,\Sigma\alpha\beta = (\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)$

$= \Sigma\alpha^2\beta + 3\alpha\beta\gamma$

$= \Sigma\alpha\,\Sigma\alpha\beta - 3\alpha\beta\gamma$

$= (-p)(q) - 3(-r) = -pq + 3r$

$= 3r - pq.$

(iv) We have, $\sum\alpha^3\beta = \alpha^3\beta + \alpha^3\gamma + \beta^3\alpha + \beta^3\gamma + \gamma^3\alpha + \gamma^3\beta$

Now, $\sum\alpha^2 \sum\alpha\beta = (\alpha^2+\beta^2+\gamma^2)(\alpha\beta+\beta\gamma+\gamma\alpha)$

$= (\alpha^3\beta + \alpha^2\beta\gamma + \alpha^3\gamma) + (\beta^3\alpha + \beta^3\gamma + \beta^2\gamma\alpha)$
$+ (\gamma^2\alpha\beta + \gamma^3\beta + \gamma^3\alpha)$

$= (\alpha^3\beta + \alpha^3\gamma + \beta^3\alpha + \beta^3\gamma + \gamma^3\alpha + \gamma^3\beta) + (\alpha^2\beta\gamma + \beta^2\gamma\alpha + \gamma^2\alpha\beta$

$= \sum\alpha^3\beta + \alpha\beta\gamma(\alpha+\beta+\gamma)$

$\Rightarrow \sum\alpha^3\beta = \sum\alpha^2 \cdot \sum\alpha\beta - \alpha\beta\gamma\sum\alpha = (p^2-2q)q - pr$

$= p^2q - 2q^2 - pr$

(v) $\sum\alpha\sum\alpha^2 = (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2) = \sum\alpha^3 + \sum\alpha^2\beta$

$\therefore \sum\alpha^3 = \sum\alpha + \sum\alpha^2 - \sum\alpha^2\beta = (-p)(p^2-2q) - (3r-pq)$ from (i) & (iii)

$= -p^3 + 2pq - 3r + pq$

$= 3pq - 3r - p^3$

(vi) $\sum\alpha^4 = \alpha^4+\beta^4+\gamma^4 = (\alpha^2+\beta^2+\gamma^2)^2 - 2(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2)$

$= (p^2-2q)^2 - 2(q^2-2pr)$ by (i) & (ii)

$= p^4 - 4p^2q + 4q^2 - 2q^2 + 4pr = p^4 - 4p^2q + 2q^2 + 4pr$

(vii) $\sum\alpha\beta\sum\alpha^2\beta^2 = (\alpha\beta+\beta\gamma+\gamma\alpha)(\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2)$

$= \sum\alpha^3\beta^3 + \sum\alpha^3\beta^2\gamma = \sum\alpha^3\beta^3 + \alpha\beta\gamma\sum\alpha^2\beta$

$\therefore \sum\alpha^3\beta^3 = \sum\alpha\beta\sum\alpha^2\beta^2 - \alpha\beta\gamma\sum\alpha^2\beta$

$= q(q^2-2pr) - (-r)(3r-pq)$ from (ii) & (iii)

$= q^3 - 2pqr + 3r^2 - pqr = q^3 - 3pqr + 3r^2$

(viii) $\sum\frac{\beta^2+\gamma^2}{\beta\gamma} = \frac{\beta^2+\gamma^2}{\beta\gamma} + \frac{\gamma^2+\alpha^2}{\gamma\alpha} + \frac{\alpha^2+\beta^2}{\alpha\beta}$

$= \frac{\alpha(\beta^2+\gamma^2) + \beta(\gamma^2+\alpha^2) + \gamma(\alpha^2+\beta^2)}{\alpha\beta\gamma}$

$= \frac{\sum\alpha^2\beta}{\alpha\beta\gamma} = \frac{3r-pq}{-r}$ from (iii)

$= \frac{pq-3r}{r}$

Ex- If α, β, γ be the roots of the eqn $x^3+3x+9=0$, Find the value of $\alpha^9+\beta^9+\gamma^9$.

Soln Here $S_1 = \alpha+\beta+\gamma = 0$, $S_2 = p^2-2q = 0-2\times3 = -6$

Now putting $x=\alpha, \beta, \gamma$ in the eqn and adding, we get

$S_3+3S_1+27=0 \Rightarrow S_3 = -27$ (since $S_1=0$)

Again $x^3 = -3(x+3)$ ---- ①

cubing both side of ①, we get

$x^9 = -27(x+3)^3 = -27(x^3+9x^2+27x+27)$ ---- ②

Putting $x=\alpha, \beta, \gamma$ in ② and then adding, we get

$S_9 = -27(S_3+9S_2+27S_1+81)$

$= -27(-27+9(-6)+27\cdot 0+81)$

$= -27(-27-54+81) = -27\times 0 = \underline{0}$

Ex- Solve the eqn $x^4-10x^3+26x^2-10x+1=0$

Soln The given eqn is a reciprocal eqn

Dividing throughout by x^2, it can be written as

$\left(x^2+\frac{1}{x^2}\right) - 10\left(x+\frac{1}{x}\right) + 26 = 0$ ---- ①

Put $x+\frac{1}{x} = y$ Then $x^2+\frac{1}{x^2}+2 = y^2 \Rightarrow x^2+\frac{1}{x^2} = y^2-2$

Hence ① becomes $(y^2-2)-10y+26=0 \Rightarrow y^2-10y+24=0$

$\Rightarrow (y-4)(y-6)=0 \Rightarrow y=4,6 \Rightarrow x+\frac{1}{x}=4$ ---- ②

and $x+1/x = 6$ ---- ③

Solving ②, we get $x^2+1=4x \Rightarrow x^2-4x+1=0$

$\Rightarrow x = \frac{4\pm\sqrt{16-4}}{2} = \frac{4\pm 2\sqrt{3}}{2} = 2\pm\sqrt{3}$

Again Solving ③, we get $x^2+1=6x \Rightarrow x^2-6x+1=0$

$\Rightarrow x = \frac{6\pm\sqrt{36-4}}{2} = \frac{6\pm\sqrt{32}}{2} = \frac{6\pm 4\sqrt{2}}{2} = 3\pm 2\sqrt{2}$

Hence the roots of the eqn are $\underline{2\pm\sqrt{3},\ 3\pm 2\sqrt{2}}$